

# Arbitrage and Control Problems in Finance. Presentation.

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## 1 Introduction

The theory of asset pricing takes its roots in the Arrow-Debreu model (see, for instance, Debreu 1959, Chap. 7), the Black and Scholes (1973) formula, and the Cox and Ross (1976) linear pricing model. This theory and its link to arbitrage has been formalized in a general framework by Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Duffie and Huang (1986). In these models, security markets are assumed to be frictionless: securities can be sold short in unlimited amounts, the borrowing and lending rates are equal, and there is no transaction cost. The main result is that the price process of traded securities is arbitrage free if and only if there exists some equivalent probability measure that transforms it into a martingale, when normalized by the numeraire. Contingent claims can then be priced by taking the expected value of their (normalized) payoff with respect to any equivalent martingale measure. If this value is unique, the claim is said to be priced by arbitrage and it can be perfectly hedged (i.e. duplicated) by dynamic trading. When the markets are dynamically complete, there is only one such a *martingale-probability measure* and any contingent claim is priced by arbitrage. The *weight* of each state of the world for this probability measure can be interpreted as the state price of the economy (the prices of \$1 tomorrow in that state of the world) as well as the marginal utilities (for consumption in that state of the world) of rational agents maximizing their expected utility.

When there are frictions, including dynamic market incompleteness, the characterization of the no-arbitrage condition is no more equivalent to the existence of a unique equivalent martingale measure. More precisely, for each

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kind of imperfection, the *equivalent martingale-measure condition* is replaced by a weaker one : *equivalent supermartingale-measure condition*, *equivalent submartingale-measure condition*, *absolutely continuous martingale-measure condition*, etc. Besides, we generally have, more than one measure satisfying these conditions. Furthermore, when there are frictions, even if a contingent claim can be duplicated by dynamic trading, it is not necessarily possible to price it by arbitrage. However arbitrage bounds can be computed, for arbitrary contingent claims, taking the expected value of their (normalized) payoff with respect to all the measures that characterize the absence of arbitrage opportunities. These bounds are the minimum amount it costs to hedge the claim and the maximum amount that can be borrowed against it using dynamic strategies. These are the tightest bounds that can be inferred on the price of a contingent claim without knowing the agent's preferences. The determination of these bounds in a dynamic setting leads to a maximization (and/or minimization) program, and, in a dynamic setting, is often transformed into a stochastic optimal control problem. The main assumption in these models is, in fact, a necessary condition for the existence of an equilibrium: the no-arbitrage condition. These preference-free theories give results of great generality without specifying the equilibrium in its full details.

Another important class of valuation theories makes assumptions on preferences and derives more specific pricing restrictions than the preference-free theory does, even in the presence of imperfections. The price of a given contingent claim, for these theories, is just the expected value of its (normalized) terminal payoff with respect to a probability measure, whose density is proportional to the marginal utility (for consumption) of the considered agent. From a mathematical point of view, starting with a given utility function, the problem is to write the first order conditions of the agent's utility maximization program, taking into account the potential imperfections in the description of the budget constraints and/or of the strategies in order to characterize the marginal utility for consumption at the final date. In a multi-period setting, this maximization problem is a stochastic optimal control problem. The main advantage of this approach is that it leads to a unique price for a given contingent claim. The main drawback is that this price depends on the choice of the utility function and on the agent endowment. If the utility function belongs to a given connected class of utility functions, we will obtain an interval of possible prices for that claim. More specifically, if the considered class is the set of all von Neumann-Morgenstern (VNM) increasing and concave utility functions, the set of possible prices is exactly the set obtained with the arbitrage approach as shown by Jouini and Kallal (1999). The unique way to obtain tighter bounds with the utility-maximization approach seems then to consider specific functions, or specific sets of functions, smaller than the set of all the VNM ones.

In fact, there is an interesting link between the two approaches. Since the arbitrage upper bound for a given contingent claim is equal to the minimum amount it costs to hedge it, taking the market frictions into account, the agent's problem (maximization of the utility provided by the terminal payoff among the strategies satisfying a given dynamic budget constraint) can be transformed

into a static problem where we maximize the utility among the set of contingent claims satisfying a budget constraint where the classical price functional is replaced by the *arbitrage upper bound* functional. If we have, for instance, an explicit formula for the arbitrage upper bound, it suffices then to solve a static maximization problem instead of the initial stochastic dynamic control problem. The characterization of the no-arbitrage assumption is therefore crucial in order to solve the contingent claim pricing problem as well as to solve the individual utility maximization problem of each agent in the economy. The last step, if we want to explore all the implications of the Arrow-Debreu model in this financial setting is then to write the equilibrium conditions in order to ensure that all the individual solutions are "compatible".

## 2 The no-arbitrage condition

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $(\mathcal{F}_t)_{t \geq 0}$  be a filtration which models our information structure. This filtration is supposed to satisfy the usual conditions, i.e. the filtration is right continuous and  $\mathcal{F}_0$  contains all negligible sets (if  $B \subset A \in \mathcal{F}$  and  $\mathbb{P}(A) = 0$  then  $B \in \mathcal{F}_0$ ). We also suppose that the sigma-algebra  $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$ , and we consider a real valued semimartingale  $S$ , which models the price process for the marketed claims. In the next, we will denote by  $\mathbb{R}$  real line and by  $\mathbb{R}_+$  the set of nonnegative ones. Let us define, as in Delbaen and Schachermayer (1994), an admissible strategy as follows:

**Definition 1** *Let  $a$  be a positive real number. An  $S$ -integrable predictable process  $H$ , is called  $a$ -admissible if  $H(0) = 0$ , and  $(H \cdot S) \geq -a$  (i.e. for all  $t \in \mathbb{R}^+$ , the stochastic integral  $(H \cdot S)_t \geq -a$  almost everywhere).  $H$  is called admissible if it is admissible for some  $a \in \mathbb{R}^+$ .*

This admissibility condition can be interpreted as a *bounded losses* condition for strategies with a zero initial investment.

We consider, as in Stricker (1990), the convex cone  $K_0$  in the space  $L_0$  of equivalent classes of measurable functions, defined up to equality almost everywhere, given by

$$K_0 = \left\{ (H \cdot S)_\infty \mid H \text{ admissible and } (H \cdot S)_\infty = \lim_{t \rightarrow \infty} (H \cdot S)_t \text{ exists a.s.} \right\}.$$

The set  $K_0$  is then the set of all terminal payoffs obtained through some admissible strategy.

**Definition 2** *We say that the semi-martingale  $S$  satisfies the condition of no-arbitrage, (NA), if  $K_0 \cap L_0^+ = \{0\}$*

Since  $K_0$  represents the set of all admissible terminal payoffs the no-arbitrage condition amounts to say that it is impossible to obtain a non-negative, non-zero payoffs with a zero initial investment.

Assume that it is possible to separate  $K_0$  and  $L_0^+$  in the sense that there exists a non-zero linear functional  $f$  and a real number  $c$  such that  $f(K_0) \leq c$  and

$f(L_0^+) > c$  and assume that the separating functional  $f$  admits a representation as an expectation operator with respect to a probability measure  $Q$ , then under the (NA) condition,  $Q$  is equivalent to  $P$  and  $E_Q[f] \leq 0$  for each  $f$  in  $K_0$ . In particular, for each  $s < t$ ,  $B \in F_s$ ,  $\alpha \in \mathbb{R}$ , we have  $\alpha(S_t - S_s)1_B \in K_0$ , therefore,  $E_Q[(S_t - S_s)1_B] = 0$ , and  $Q$  is a martingale measure for  $S$ .

Unfortunately, the (NA) condition is seldom sufficient to apply a separation theorem. In the case where  $S$  is locally bounded we have the following:

**Theorem 1 (Delbaen and Schachermayer, 1994)** *Let  $S$  be a locally bounded real valued semi-martingale. There is an equivalent local martingale measure  $Q$  for  $S$  if and only if  $(K_0 - L_+^0) \cap L^\infty \cap L_+^\infty = \{0\}$  (where  $L^\infty$  is the space of bounded measurable functions and where the closure is taken with respect to the norm-topology of  $L^\infty$ ).*

This last condition is called No Free-Lunch with Vanishing Risk (NFLVR) and deals with sequences of strategies such that the negative parts of their terminal payoff tends to zero uniformly instead of strategies with non-negative terminal payoff as in the (NA) condition. Remark that the main difference between these two conditions lies in the fact that we have to consider a closure. Indeed, the condition  $[(K_0 - L_+^0) \cap L^\infty] \cap L_+^\infty = \{0\}$  is weaker than  $(K_0 - L_+^0) \cap L_+^\infty = \{0\}$ , which is equivalent to the (NA) condition  $K_0 \cap L_+^\infty = \{0\}$ .

If, in the previous condition, we replace the norm-topology closure by the  $\sigma(L^\infty, L^1)$  topology closure (where  $L^1$  is the space of all integrable  $\mathcal{F}$ -measurable functions) we obtain a version of the No Free-Lunch (NFL) condition introduced by Kreps (1981), and the existence of an equivalent local martingale measure is obtained for a bounded càdlàg, and adapted process  $S$ .

Other intermediary concepts, as the No Free-Lunch with Bounded Risk (NFLBR) condition (where the closure is defined as the set of weak\*-limits and where the negative parts of the terminal payoffs tend to zero in probability and remains uniformly bounded), have been introduced in the literature and permit to obtain results similar to the previous theorem in different contexts: finite time set and (NA) condition with Dalang et al. (1989) (see Schachermayer (1992), Kabanov and Kramkov (1994a) and Rogers (1995) for elementary proofs), infinite but discrete time set with Schachermayer (1994), continuous and bounded processes in continuous time with Delbaen (1992). Harrison and Kreps (1979) and Harrison and Pliska (1981) used the concept of simple strategies. Kreps (1981) used a concept of "no free-lunch" involving the convergence of nets or generalised sequences. Duffie and Huang (1986) and Stricker (1990) used  $L^p$  convergence. Lakner (1993) used convergence in Orlicz spaces. Furthermore, it appears that the class of semi-martingales is the most general one compatible with this kind of results. Indeed, from the work of Föllmer and Schweizer (1991) and Ansel and Stricker (1993) we know that No Free-Lunch conditions imply, in some sense, that  $S$  is a semi-martingale. Conversely, the existence of an equivalent-martingale measure for  $S$  implies, by Girsanov's theorem, that  $S$  is a semi-martingale.

As in Harrison and Kreps (1979) and Kreps (1981) and for a given contingent claim, we define the arbitrage pricing interval as the set of all the prices that are

compatible with the No Free-Lunch condition, i.e. introducing this contingent claim at one of these prices does not create free-lunches. In all the mentioned papers, it appears that this set is equal to the set of expected values of the considered claim terminal payoff with respect to all the probability-measures which characterize the absence of free-lunch.

When there is no imperfection, the cash-flow space  $\Phi$  can be identified with  $K_0$ . In case of short-sale constraints, it can be identified with

$$\left\{ (H \cdot S)_\infty \mid H \text{ admissible, } H \geq 0 \text{ and } (H \cdot S)_\infty = \lim_{t \rightarrow \infty} (H \cdot S)_t \text{ exists a.s.} \right\}$$

and when there are short-selling costs, it can be identified with

$$\left\{ (H \cdot S - H' \cdot S')_\infty \mid \begin{array}{l} (H, H') \text{ admissible, } (H, H') \geq 0 \\ \text{and } (H \cdot S - H' \cdot S')_\infty \text{ exists a.s.} \end{array} \right\}$$

where  $S$  (resp.  $S'$ ) models the long (resp. short) position returns. When there are transaction costs,  $\Phi$  can be identified with

$$\{(H \cdot S)_\infty - \lambda V((H \cdot S)) \mid H \text{ admissible, and } (H \cdot S)_\infty \text{ exists a.s.}\}$$

where  $V$  represents the total variation of  $(H \cdot S)$  and  $\lambda$  the magnitude of the transaction costs. Jouini and Kallal (1995a and b) characterized first the absence of arbitrage opportunities in these different situations. Other contributions on related subjects are due to Kabanov and Kramkov (1994b), Shirakawa and Konno (1995), Kusuoka (1995), Cvitanic and Karatzas (1996), Cvitanic, Pham and Touzi (1999), Kabanov (1999). The differences between all these references are in the choice of the topology (or no topology) in order to define the concept of free-lunch, the choice of a space of admissible strategies (discrete strategies, simple strategies,...) and finally the choice of possible imperfections (or no imperfection). This choice is summarized by the choice of a convex cone contained in  $K_0 - L_+^0$  instead of  $K_0 - L_+^0$  itself in order to model the opportunity set. In this context, it is *easier* to find a separating hyperplane between that set (or its closure with respect to some topology) and  $L_+^\infty$ . Jouini and Kallal (1999) extended all the arbitrage, viability and equilibrium classical results to that setting mainly by assuming that the opportunity set is a convex cone (or even a convex set) and the pricing rule is sublinear. In this issue, Kabanov and Stricker (2001) propose a generalization of Jouini and Kallal's (1995a) result to the important case of a multi-asset market model where the transaction costs are defined for each kind of transaction between any pair of assets. They use the geometric formalism developed previously by Kabanov (1999) and they characterize the absence of arbitrage opportunities in terms of martingale-like measures. Their result is established in a discrete time and finite set of states of the world framework and they only deal with arbitrages and not with free-lunches.

In order to take a large set of possible frictions into account Carassus and Jouini (1997,1998,2000) in discrete time or in a deterministic setting, Jouini and Napp (2000) and Jouini, Napp and Schachermayer (2000) in continuous

time propose to deal directly with the space of possible cash-flows instead of the space of terminal payoffs and they provide a characterization of the No Free-Lunch assumption in terms of the existence of a *separating* functional. Napp (2001) develops an arbitrage pricing theory and a super-replication concept in this cash-flow space.

However, all these results are obtained under a convexity condition on the space of attainable payoffs. This last assumption is not satisfied in economies with fixed costs, i.e. with transaction costs which are not proportional to the size of the transactions. In this framework, the terminal payoff of a strategy  $H$  is  $(H \cdot S)_\infty + c(H)$ , where  $c$  is a bounded non-linear function of the strategy  $H$ , instead of  $(H \cdot S)_\infty$  as in the classical case. Therefore, it is easy to understand that large scale transactions will *kill* the transaction cost effect and that the characterization of the no-arbitrage condition should be an asymptotic version of the classical one. Jouini, Kallal and Napp (2001) prove that this characterization is in terms of absolutely continuous martingale measures and show that the existence of such a measure is necessary but not sufficient and that we need the existence of a family of such measures each one associated with a given date and a given event at that date in order to characterize the absence of free-lunches.

### 3 The utility maximization problem

Before Merton's (1969) paper, most models of portfolio selection only considered one period. Furthermore, the investment decision by households was viewed in two parts : (a) the "consumption-saving" choice where the individual decides how much income and wealth to allocate into current consumption and how much to save for future consumption; and (b) the "portfolio-selection" choice where the investor decides how to allocate savings among the available investment opportunities. Merton (1969) examined "*the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous-time model where his income is generated by returns on assets and these returns or instantaneous 'growth rates' are stochastic*".

The original analysis of Merton's model is based on the Hamilton-Jacobi-Bellman equation and requires an underlying Markov state process. After the papers of Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), and their characterization of the no-arbitrage assumption in terms of the existence of martingale-measures, Pliska (1986), Cox and Huang (1989, 1991) and Karatzas, Lehoczky and Shreve (1987) used this methodology in order to analyze this consumption-investment problem. This new approach is based on duality arguments and permits to transform the initial dynamic problem into a static one and to solve it without assuming any "Markov" condition.

Let us now introduce the main results related to this problem.

Let  $(\Omega, F, P)$  be a fixed probability space and  $\mathbf{T}$  denote the interval  $[0, T]$ , on which we are going to treat our problem :  $T$  corresponds to the terminal date for all economic activity under consideration. All processes that we shall encounter in this section are defined on  $\mathbf{T}$ .

We consider a market consisting in one bond and  $N$  assets. More precisely, the primitive market model is the same as in Karatzas (1989), except that we consider here dividends paying assets.

We adopt a model for the market consisting of one bond with price at time  $t$  denoted by  $S_t^0$  satisfying the differential equations

$$dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1, \quad (1)$$

and  $N$  stocks with prices at time  $t$  denoted by the  $N$ -dimensional vector<sup>1</sup>  $S_t$  satisfying

$$dS_t = S_t [(b_t - \delta_t) dt + \sigma_t dW_t], \quad S_0 = 1. \quad (2)$$

Here,  $W = \{(W_t^1, \dots, W_t^N)^*; t \in \mathbf{T}\}$  is a  $N$ -dimensional Brownian motion on a probability space  $(\Omega, \mathcal{F}, P)$  and  $(F_t)_{t \in \mathbf{T}}$  denotes the  $P$ -augmentation of the natural filtration generated by  $W$ . We assume that the sample paths of  $W$  specify completely all the distinguishable events, which mathematically entail  $F_T = \mathcal{F}$ . Since standard Brownian motions start from zero with probability one,  $F_0$  is trivial. We will denote by  $L_d^2(\mathbf{T})$  the set of  $(F_t)_{t \in \mathbf{T}}$ -progressively measurable processes  $\{\Psi_t; t \in \mathbf{T}\}$  taking values in  $R^d$  such that

$$\int_0^T \|\Psi_t\|^2 dt < \infty \quad \text{a.s. } P.$$

**Assumption A1** *The real-valued interest rate process  $\{r_t; t \in \mathbf{T}\}$ , the  $N$ -dimensional process  $\{b_t; t \in \mathbf{T}\}$ , the  $N$ -dimensional dividend yield process  $\{\delta_t; t \in \mathbf{T}\}$  as well as the volatility  $(N \times N)$ -matrix-valued process  $\{\sigma_t; t \in \mathbf{T}\}$  are supposed to be progressively measurable with respect to  $(F_t)_{t \in \mathbf{T}}$  and bounded uniformly in  $(t, \omega)$  in  $\mathbf{T} \times \Omega$ .*

Under this assumption<sup>2</sup>, Equation (2) admits a unique real-valued,  $(F_t)_{t \in \mathbf{T}}$ -adapted, continuous solution  $\{S_t; t \in \mathbf{T}\}$ , satisfying  $E[\sup_{t \in \mathbf{T}} S_t^2] < \infty$ .

**Assumption A2** *For all  $t$  in  $\mathbf{T}$ , the volatility matrix  $\sigma_t$  has full rank  $N$  and the norm of  $(\sigma_t)^{-1}$  is uniformly bounded.*

Therefore, a  $N$ -dimensional process  $\theta = \{\theta_t; t \in \mathbf{T}\}$  can be defined by<sup>3</sup>:

$$\theta_t \equiv (\sigma_t)^{-1} [(b_t - r_t 1_N)] \quad P \text{ a.s., } 0 \leq t \leq T.$$

With the above assumptions,  $\theta$  is  $(F_t)_{t \in \mathbf{T}}$ -progressively measurable and uniformly bounded.

We shall also introduce the discounted price process  $\tilde{S} = \{\tilde{S}_t; t \in \mathbf{T}\}$  defined by  $\tilde{S}_t \equiv S_t \exp \int_0^t (\delta_s - r_s) ds$  for all  $t$  in  $\mathbf{T}$ . Using Itô's Lemma, we easily get that  $\tilde{S}$  is the unique solution of the following stochastic differential equation:

$$d\tilde{S}_t = \tilde{S}_t [(b_t - r_t) dt + \sigma_t dW_t] = \tilde{S}_t \sigma_t [\theta_t dt + dW_t], \quad \tilde{S}_0 = 1.$$

<sup>1</sup>All vectors are column vectors and transposition is denoted by the superscript  $*$ . We denote by  $\|Z\|^2$  the nonnegative real number  $\sum_{i=1}^n (Z^i)^2$ .

<sup>2</sup>See for instance Karatzas-Shreve [1988]

<sup>3</sup>As usual,  $1_d$  denotes the  $d$ -dimensional vector whose component are equal to one.

Notice that assets prices can fluctuate in an almost arbitrary, not necessarily Markovian way.

We know that in such a model, there exists a unique equivalent probability measure  $\bar{P}$  defined on  $(\Omega, F, P)$  that makes the full process  $\tilde{S}$  a martingale for  $(F_t)_{t \in \mathbf{T}}$ . It is given by<sup>4</sup>

$$d\bar{P}/dP = \mathcal{E}_T(-\theta).$$

We then have<sup>5</sup>  $d\tilde{S} = \text{diag}[\tilde{S}_t] \sigma_t dW_t^{\bar{P}}$ , where  $\{W_t^{\bar{P}}; t \in \mathbf{T}\}$  is the  $\bar{P}$ -Brownian motion for  $(F_t)_{t \in \mathbf{T}}$  defined by  $W_t^{\bar{P}} \equiv W_t + \int_0^t \theta_s ds$  for all  $t$  in  $\mathbf{T}$  (see Girsanov's Theorem). We shall denote in the following, the martingale process  $\left\{ \mathbb{E} \left[ \frac{d\bar{P}}{dP} \mid F_t \right]; t \in \mathbf{T} \right\}$  by  $\bar{M} = \{\bar{M}_t; t \in \mathbf{T}\}$ .

In the context of the above market-model, consider an agent who starts out with an initial capital  $x$  and can decide of the amounts  $\pi_t = (\pi_t^1, \dots, \pi_t^N)$  that he invests at time  $t$  in the different assets, and of the rate  $c_t$  at which he withdraws funds for consumption. Assuming that at each time  $t$ , sales and dividends must finance purchases and consumption, the corresponding wealth process, denoted by  $X_t^{x;\pi,c}$ , satisfies the following stochastic differential equation

$$dX_t^{x;\pi,c} = \sum_{i=0}^N \frac{\pi_t^i}{S_t^i} (dS_t^i + \delta_t^i S_t^i dt) - c_t dt \quad (1^{st} \text{ Self-financing condition})$$

$$X_0^{x;\pi,c} = x \quad (3)$$

which can be rewritten

$$dX_t^{x;\pi,c} = [r_t X_t^{\pi,c} - c_t] dt + (\pi_t)^* \sigma_t dW_t^{\bar{P}} \quad (2^{nd} \text{ Self-financing condition})$$

$$X_0^{x;\pi,c} = x \quad (4)$$

The set of investment-consumption strategies  $(\pi, c)$  satisfying the previous self-financing condition and the following no-bankruptcy condition is called the admissible strategies set and denoted by  $\mathbf{A}(x)$  :

$$\forall t \in \mathbf{T}, X_t^{x;\pi,c} \geq 0, \quad (\text{No-bankruptcy condition})$$

This last condition amounts to saying that at each time  $t$ , the investor must be able to cover his debts -see e.g. Karatzas-Lehoczky-Shreve (1987) or Duffie (1992) where the same assumption is made.

<sup>4</sup>For any  $R^d$ -valued process  $\Psi = \{\Psi_t; t \in \mathbf{T}\}$  in  $L_d^2(\mathbf{T})$ , let the real-valued process  $\mathcal{E}(\Psi) = \{\mathcal{E}_t(\Psi); t \in \mathbf{T}\}$  denote the exponential local martingale given for each  $t$  in  $\mathbf{T}$  by  $\mathcal{E}_t(\Psi) = \exp \left\{ \int_0^t (\Psi_s)^* dW_s - 1/2 \int_0^t \|\Psi_s\|^2 ds \right\}$ .

<sup>5</sup>If  $Z = (Z^1, \dots, Z^n)$  denotes a vector in  $R^n$ , then  $\text{diag} Z$  denotes the  $(n \times n)$  diagonal matrix whose diagonal entries are the components of  $Z$ .



Under the self-financing condition, the process

$$Y \equiv \left\{ \exp \left( - \int_0^t r_s ds \right) X_t^{x;\pi,c} + \int_0^t \exp \left( - \int_0^s r_u du \right) c_s ds; t \in \mathbf{T} \right\}$$

consisting in the current discounted wealth plus the total discounted consumption is a  $\bar{P}$ -supermartingale (Fatou's Lemma). It is then easy to see that the market excludes any arbitrage opportunity which turns out to be characterized in our context by the existence of a pair  $(\pi, c)$  in  $\mathbf{A}(0)$  such that  $P[X_T^{0;\pi,c} > 0] > 0$ .

Let  $\mathbf{B}$  denote the set of pairs  $(X, c)$  where  $c$  is an adapted nonnegative consumption rate process and  $X$  is a nonnegative  $F_T$ -measurable random variable describing the terminal wealth. An agent is represented by a utility function  $U : \mathbf{B} \rightarrow \mathbb{R}$ , given by

$$U(c, X) = E \left[ \int_0^T u(t, c_t) dt + V(X) \right],$$

where  $u$  and  $V$  satisfy the following assumption.

**Assumption  $\mathcal{U}$**  *The function  $V : R_+ \rightarrow R$  is<sup>6</sup>  $C^1$  on  $(0, \infty)$ , strictly increasing, strictly concave and satisfies Inada conditions ( i.e.,  $\inf_x V'_j(x) = 0$  and  $\sup_x V'_j(x) = +\infty$ ). The function  $u_j : \mathbf{T} \times R_+ \rightarrow R$  is  $C^{0,1}$  on  $\mathbf{T} \times R_+$ , strictly increasing, strictly concave and satisfies Inada conditions.*

Under Assumption  $\mathcal{U}$ , we shall denote by  $u_c(t, \cdot)$  the derivative of  $u(t, \cdot)$  and by  $I_u(t, \cdot)$  the inverse function of  $u_c(t, \cdot)$ , which is a strictly decreasing continuous function on  $(0, \infty)$  in  $(0, \infty)$ . We shall also denote by  $I_V$  the inverse function of  $V'(\cdot)$ .

The considered agent has an initial endowment  $x$  and tries to maximize his utility  $U(c, X)$  on both his consumption over the time-interval  $\mathbf{T}$  and his terminal wealth. The optimal demand  $(c)^*$  of the agent in the consumption commodity as well as his optimal portfolio  $(\pi)^*$  are determined by the optimization problem

$$\sup_{(\pi, c) \in \mathbf{A}(x)} U(c, X_T^{x;\pi,c}) = \sup_{(X, c) \in \mathbf{B}} U(c, X).$$

Adapting the proofs of Duffie (1994) and Karatzas (1989), we get that

**Proposition 2** *A pair  $(c^*, X^*)$  in  $\mathbf{B}$  is optimal for an agent with an initial endowment  $x$  if and only if there exists a constant  $\gamma^* > 0$  such that*

$$c^*(t) = I_u \left( t, \frac{1}{\gamma^*} \exp \left( - \int_0^t r_s ds \right) \bar{M}_t \right) \quad 0 \leq t \leq T \quad a.s. \quad P \quad (5)$$

$$X^* = I_V \left( \exp \left( - \int_0^T r_s ds \right) \bar{M}_T \right) \quad (6)$$

<sup>6</sup>As usual, a function  $F : \mathbf{T} \times R \rightarrow R$  is said to be of class  $C^{m,n}$  if the  $m$ -th derivative of  $F(\cdot, x) : \mathbf{T} \rightarrow R$  and the  $n$ -th derivative of  $F(t, \cdot) : R \rightarrow R$  exist and are continuous.

$$\mathbb{E} \left[ \int_0^T \exp \left( - \int_0^t r_s ds \right) \bar{M}_t c^*(t) dt + \exp \left( - \int_0^T r_s ds \right) \bar{M}_T X^* \right] = x. \quad (7)$$

This last proposition permits to solve explicitly the agent's optimization program. Huang and Pagès (1992) extended this methodology to the infinite horizon framework. Karatzas, Lehoczky, Sethi and Shreve (1986) provided explicit computations in that framework assuming constant coefficients in the price evolution equations. He and Pearson (1991 a and b) and Karatzas, Lehoczky, Shreve and Xu (1991) extended the methodology to incomplete markets and proved that the optimal investment/consumption plan is given, as in the classical case, by the inverse of the marginal utility evaluated at the random variable which is optimal for a well-defined dual problem. Cvitanič and Karatzas (1992) used the same approach in order to solve the problem when there is convex constraints on the strategies (short-sales constraints, borrowing constraints,...). Fleming and Zariphopoulou (1991) solved the problem assuming different borrowing and lending rates. Cuoco (1997) and El Karoui and Jeanblanc-Picqué (1997) considered random endowment streams. Cvitanič and Ma (1996) and Cuoco and Cvitanič (1998) generalized these results in the context of a "large investor". In that context, the strategy of the investor has a direct nonlinear impact on the price dynamics. The main technique in all these references consists in embedding the original problem into a family of perfect (linear) "fictitious" markets, where security prices dynamics are modified and agents receive an additional stochastic "endowment" reflecting the nonlinearity in the market price of risk. The fictitious markets are designed in such a way that the optimal policy in one of them coincides with that in the actual, nonlinear market.

Using the partial differential equations (PDE) approach, Dumas and Luciano (1989) first formulated the problem in the presence of transaction costs. The main contributions in this context are Davis and Norman (1990), Fleming, Grossman, Vila and Zariphopoulou (1990) and Shreve and Soner (1994).

When there are imperfections, the utility maximization approach can be used in order to provide pricing formulas for new contingent claims. There are mainly two methods. The first one, initiated by Hodges and Neuberger (1989) in the transaction costs setting, consists in using the marginal utility of the considered agent at his optimal consumption-investment plan as a state-price density. The second method, initiated by Davis (1994) consists in comparing the optimal utility levels  $\mathbb{V}^x$  with a deterministic initial endowment  $x$  and  $\mathbb{V}^*$  with a stochastic endowment equal to the payoff of the considered claim. The "fair" price  $x$  is then defined by the equation

$$\mathbb{V}^x = \mathbb{V}^*.$$

Papers along these lines include Constantinides (1986), Panas (1993), Davis, Panas and Zariphopoulou (1993), Davis and Panas (1994), Davis and Zariphopoulou (1995), Cvitanic and Karatzas (1996), Constantinides and Zariphopoulou (1997) and Barles and Soner (1998).

In this issue, Cvitanic and Wang (2001) show that the martingale/duality approach adopted in the frictionless model works also in the transaction costs framework and prove that the optimal terminal wealth is given as the inverse of marginal utility evaluated at the random variable which is optimal for an appropriately defined dual problem. They prove the existence of a solution for this dual problem and doing so they resolve a question left open by Cvitanic and Karatzas (1996). A similar problem is studied by Deelstra, Pham and Touzi (2000) where the utility functions are defined on  $\mathbb{R}^d$  instead of  $\mathbb{R}$ .

Framstad, Øksendal and Sulem (2001) considers also the transaction costs framework but in a jump diffusion market. Using a viscosity solution approach, they show that the solution of the problem in that context has the same form as in the pure diffusion case : there is a no-transaction cone such that it is optimal to make no transactions as long as the wealth position remains in that cone and to trade on the boundary. Bellamy (2001) solves the same problem but assuming market incompleteness instead of the presence of transaction costs and using a Hamilton-Jacobi-Bellman (HJB) approach.

Using filtering techniques, Lakner (1995) considers utility maximization problems where the agent must estimate the mean rate of return of the assets. In this issue, Dokuchaev and Zhou (2001) considers the case where the stock appreciation rates are not observable and where the strategies depend only on the known distribution of these rates and on the current prices. Furthermore, they use general utility/loss functions (including mean-variance criteria and goal achieving problems) and they consider some lower and upper constraints on the terminal wealth. The problem is solved by means of backward stochastic differential equations as well as a dual formulation.

Finally, we want to mention another family of optimization problems related to the contingent claims pricing : the hedging problems. These problems are not represented in this special issue but are studied by many papers in the recent literature. The main problem in all these paper is to compute the hedging price of a given contingent claim with respect to a given hedging criterion. If we assume that the agents want to minimize the downside risk, then the hedging price is equal to the super-replication price and its computation leads to solve a stochastic-control-based problem as in the pioneering paper of El Karoui and Quenez (1991, 1995). If we assume that the agents want to minimize the quadratic risk then we have to solve the mean-variance hedging problem and we refer to Föllmer and Schweizer (1991) or Schweizer (1993) for a survey about related results.

## 4 The equilibrium

Models of competitive equilibrium go back to Walras (1874). The first complete proof for the existence of an equilibrium in an economy with finitely many commodities was given by Arrow and Debreu (1954). In the chapter 7 of Debreu (1959), the author explains how this model permits to take into account dynamic markets with uncertainty. Bewley (1972) studied the competitive equi-

librium in an infinite-dimensional commodity space, namely  $L^\infty$  and Mas-Colell (1986) generalized Bewley's results to Hausdorff locally convex, topological vector spaces under a "uniform properness condition" on the agent's preferences. Araujo and Monteiro (1989 a and b) and Duffie and Zame (1989) proved independently the existence of an equilibrium without Mas-Colell's uniform properness condition. Dana, Le Van and Magnien (1997) extended Mas Colell's result to topological locally solid Riesz spaces under a local non-satiation condition weaker than the uniform properness one. Aase (1992) and Bernis (2000) applied these results to the reinsurance markets.

All the previous models does not take explicitly into account dynamic security trading. Models where the agents achieve equilibrium allocations by trading in securities like the capital asset pricing model (CAPM) or the consumption based capital asset pricing model (CCAPM) can be found in the literature going back to Merton (1971), Cox, Ingersoll and Ross (1985), Duffie and Huang (1985), Huang (1987) and Karatzas, Lehoczky and Shreve (1990).

The link between these two approaches is made by Duffie and Huang (1985) where the authors explain how an Arrow-Debreu equilibrium can be implemented by trading in securities. This role of securities was, in fact, already recognized by Arrow (1952). The difference between the two approaches is illustrated by Cuoco (1997) where the budget constraints are associated to all the possible equilibrium prices (all the risk-neutral measures) instead of a unique budget constraint associated to the equilibrium price as in the classical general equilibrium model.

In Karatzas, Lehoczky and Shreve (1990), all agents are endowed in units of the same perishable commodity, which arrives at some time-varying random rate. Agents may consume their endowment as it arrives, they may sell some portion of it to other agents, or they may buy extra endowment from other agents. The endowment, however, cannot be stored, and agents wish to hedge the variability in their endowment process by trading with one another.

In this model all the prices are in term of a unique consumption good. When the market is complete it is equivalent to assume that the agents receive their endowment initially rather than over time. In that case and in order to have a stochastic total wealth, we assume that the consumption good is produced by the firms and distributed as dividends among the shareholders. The equilibrium condition imposes then a total consumption equal to the total supply of the consumption good and a total investment in each firm equal to the total value in term of consumption good of that firm.

With the notations of Jouini and Napp (1998), the mathematical description of the model is the following. Let  $\mathbb{E}$  be an economy with  $n$  agents indexed by  $j = 1, \dots, n$  and let us assume that the  $j^{th}$  agent has an initial wealth  $x_j$  and a utility function  $U_j : \mathbf{B} \rightarrow \mathbb{R}$  given by

$$U_j(c, X) = \mathbb{E} \left[ \int_0^T u_j(t, c_t) dt + V_j(X) \right]$$

where  $u_j$  and  $V_j$  satisfy our assumption  $\mathcal{U}$ . As previously each agent maximizes

his utility level over the set of admissible strategies. More precisely, the  $j^{th}$  agent problem is

$$\sup_{(\pi, c) \in \mathbf{A}(x_j)} U_j(c, X_T^{x_j; \pi, c}).$$

In this framework an equilibrium consists in a  $N$ -dimensional price processes  $S$  and trading-consumption choices  $(\pi_j^*; c_j^*)_{1 \leq j \leq n}$  which are optimal for the agents, i.e.

$$(\pi_j^*; c_j^*) \in \arg \max_{(\pi, c) \in \mathbf{A}(x_j)} U_j(c, X_T^{x_j; \pi, c})$$

and such that for all  $t$  in  $\mathbf{T}$ , the following market clearing conditions hold almost surely:

$$\begin{aligned} \sum_{j=1}^n (c_j^*)_t &= \delta_t \cdot 1_N \\ \sum_{j=1}^n (\pi_j^*)_t^i &= q^i S_t^i, \quad i = 0, \dots, N \\ \sum_{j=1}^n X_t^{\pi_j^*, c_j^*} &= S_t. \end{aligned}$$

where  $q^i$  is the number of firm  $i$  outstanding shares

Note that the last condition is redundant with the two previous ones by the self-financing condition.

In Karatzas, Lehoczky and Shreve (1990) it is shown that under mild conditions a unique equilibrium exists. In this issue, Chiarolla and Haussmann (2001) specializes and extends the Karatzas and al. (1990) model to a situation where the endowment streams of the agents are denominated in money, not in goods, and are not exogenous. The labor provided by the agents to a firm produces the consumable good through a production function. The agents have then to choose a consumption and a leisure levels in order to maximize their utility function. Furthermore, the firm defines the level of employment by a profit maximization program. The utility functions of the agents depend then on two control variables and the main contribution of this paper is to extend the classical one-dimensional approach to this framework. The authors provide first order necessary conditions for equilibrium, and derive from there the existence of such an equilibrium. They also solve explicitly two examples.

Basak and Croitoru (2001) exploit the equilibrium conditions in order to analyze the taxation impact on the asset prices. They consider a simple two agents model and use the "fictitious" market techniques described in the previous section in order to solve the individual utility maximization problem. The main difficulty is due to the presence of two redundant assets but with different taxation rules. The redundancy adds an extra step in the agent's problem : once he has chosen his risk exposure, he must decide how to allocate that risk

between the two securities. The authors establish general necessary conditions for equilibrium and show, in particular, that arbitrage opportunities still exist at the optimum. They characterize this "mispricing" and they provide an analysis of its equilibrium role.

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